EXISTENCE AND POSITIVITY PROPERTIES OF SOLITARY WAVES FOR A MULTICOMPONENT LONG WAVE–SHORT WAVE INTERACTION SYSTEM

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ABSTRACT. We study the existence of solitary-wave solutions and some of their properties for a general multicomponent long-wave–short-wave interaction system. The system considered here describes the nonlinear interaction of multiple short waves with a long-wave, and is of interest in plasma physics, nonlinear optics, and fluid dynamics.

1. Introduction

The long wave-short wave interaction (LSI) is an important problem in a variety of physical systems. The LSI model has been successfully applied to many different contexts of modern physics and fluid dynamics, such as studying the solitons resulting from the interactions between long ion-sound waves (ion-acoustic waves) and short Langmuir waves (plasma waves, plasmons) in a magnetized plasma [18, 31], or Alfvén-magneto-acoustic waves interactions in a cold plasma [28]. Kawahara et al. [19] have investigated the nonlinear interaction between short and long capillary-gravity waves on a liquid layer of uniform depth. For a general theory for deriving nonlinear PDEs which permit both long and short wave solutions and interact each other nonlinearly, the reader may consult [5].

In recent years there has been renewed interest in the study of nonlinear waves in multi-component LSI system. The multi-component LSI systems arise in water waves theory [15], optics [24], ferromagnetism theory [23], acoustics [26], in a bulk elastic medium [16], to name a few. In this paper we consider a general multi-component LSI system describing the interaction of multiple NLS-type short waves with a KdV-type long wave in one dimension. The nonlinear interaction between N complex short-wave field envelopes, $u_j, j = 1, 2, ..., N$, and the real long-wave field, v, can be modeled by

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the $(N+1)$-component long-wave–short-wave system

$$\begin{cases} i\partial_t u_1 + \partial_x^2 u_1 = -\alpha_1 u_1 v, \\ \dots \qquad \dots \qquad \dots \\ i\partial_t u_N + \partial_x^2 u_N = -\alpha_N u_N v, \\ \partial_t v + \partial_x \left(\gamma \partial_x^2 v + Q(v)\right) = -\partial_x \left(\beta_1 |u_1|^2 + \dots + \beta_N |u_N|^2\right), \end{cases} \tag{1.1}$$

where x and t are spatial and temporal variables, respectively, $Q = Q(v)$ is a nonlinear polynomial, and the parameters $\alpha_j, \beta_j, \gamma$ are real constants. The motivation for studying systems of the form (1.1) also come from a pioneer work of Kanna et al. [17], who set $\gamma = 0, \alpha_j = \alpha = \beta_j$, and $Q \equiv 0$ in (1.1) and proved that (1.1) can be derived from a system of multi-component coupled nonlinear Schrödinger type equations. They have also shown that the system is integrable via Painlevé test. In [22], the system (1.1) with $\gamma = 0, \alpha_j = \alpha = \beta_j, Q \equiv 0$, and $N = 2$ has been shown to be integrable by the inverse scattering transform method and the soliton solutions have been obtained. In the same case, the rogue waves of (1.1) have been reported in [11]. We also mention the paper [12] where general bright-dark multi-soliton solution has been constructed for a general multicomponent LSI system.

The mathematical study of systems of the form (1.1) with $N = 1$ and $Q(v) = v^2$, namely well-posedness results (unique existence, persistence, and continuous dependence on initial data) on the associated Cauchy problem or existence and qualitative properties of solutions, has been studied extensively over the years by many authors using both numerical and theoretical techniques (see for example, [4, 6, 9, 14, 30] and references therein). Despite some progress has been made so far using numerical and algorithms methods, many difficult questions remain open and little is known about theoretical results concerning existence and properties of solutions for $(N+1)$-systems (1.1) for $N \geq 2$. This project aims to cast a light on $(N+1)$-component long-wave–short-wave interaction system. Included in the study are existence results and several properties of travelling solitary waves for (1.1) in the case when $\beta_j = \alpha_j/2$ and $Q(v) = \tau v + \beta v^2$ with $\tau \in \mathbb{R}$ and $\beta \geq 0$.

By the travelling-wave solutions of (1.1) we mean the solutions of the form

$$\mathbb{T} = \left\{ \left(e^{i\omega t}\phi_1(x-ct), \dots, e^{i\omega t}\phi_N(x-ct), \Psi(x-ct)\right) : c, \omega \in \mathbb{R} \right\}. \tag{1.2}$$

Usually a nontrivial (i.e., not identically zero) travelling-wave solution which vanishes at $\pm\infty$ (say, ϕ_j and Ψ are in $H^1(\mathbb{R})$, the usual Sobolev space) is referred to as a solitary wave. In the case when $c = 0$ (zero travelling velocity), these solutions (1.2) are time independent which usually are referred to as standing-wave solutions or stationary-state solutions. Let $(u_1, \dots, u_N, v)$ be a solution of the form (1.2). Put $\phi_j(x) = e^{icx/2}\Phi_j(x), j = 1, \dots, N$, and substitute u_j and v into (1.1), integrate the second equation once, and evaluate the constant of integration by using the fact that Φ_j and Ψ are H^1 functions. Then, one sees that $(\Phi_1, \dots, \Phi_N, \Psi)$ must satisfy the following

system of ordinary differential equations

$$\begin{cases} \Phi_1'' - \sigma\Phi_1 = -\alpha_1\Phi_1\Psi, \\ \dots \quad \dots \quad \dots \\ \Phi_N'' - \sigma\Phi_N = -\alpha_N\Phi_N\Psi, \\ \gamma\Psi'' - c_\tau\Psi = -\frac{1}{2}\beta\Psi^2 - \frac{1}{2}\left(\alpha_1\Phi_j^2 + \dots + \alpha_N\Phi_N^2\right), \end{cases} \tag{1.3}$$

where $c_\tau = c - \tau$, $\sigma = \omega - c^2/4$, and primes denote derivatives with respect to the variable ξ. This equation will be considered the defining equation for solitary waves.

Our analysis begins with a study of the existence of solutions of the system (1.3). We prove the existence of nontrivial solutions of (1.3) with each component in H^∞ and exponential decay at infinity under the following assumptions

$$c > 0,\ \sigma > 0,\ -\infty < \tau \le c,\ \beta \ge 0,\ \gamma \ge 0,\ \alpha_j > 0 \tag{1.4}$$

(see Theorem 2.1 below for the precise statement of the result). The existence result is proved by studying a minimization problem whose minimizers, up to a constant, corresponds to solitary waves for (1.1). More precisely, let K be the functional defined for $(f_1, ..., f_N, g) \in (H^1)^{N+1}$ by

$$\mathsf{K}(f_1, \dots, f_N, g) = \int_{-\infty}^{\infty} (f_1, \dots, f_N, g) D(L_{ii}) (f_1, \dots, f_N, g)^T \, dx, \tag{1.5}$$

where $D(L_{ii})$ is the diagonal matrix with diagonal entries $L_{ii} = \sigma - \partial_{xx}$ for $i = 1, \dots, N$, and $L_{ii} = c_\tau - \gamma\partial_{xx}$ for $i = N+1$, and introduce the notation

$$F(f_1, \dots, f_N, g) = \frac{1}{3}\beta g^3 + \left(\alpha_1 f_1^2 + \alpha_2 f_2^2 + \dots + \alpha_N f_N^2\right) g.$$

For $\lambda > 0$, we shall show that the variational problem (P1) of minimizing the functional $\mathsf{K}(f_1, ..., f_N, g)$ subject to the constraint

$$\int_{-\infty}^{\infty} F(f_1, \dots, f_N, g)(x) \, dx = \lambda$$

always has a non-empty solution set provided that (1.4) holds. The key idea in establishing the existence of minimizers here is to apply Lions' concentration compactness lemma (Lemma 2.5) to a minimizing sequence of the problem (P1) and extract a subsequence which is tight. The method of concentration compactness then implies that this subsequence, when its terms are suitably translated, converges strongly in $(H^1)^{N+1}$ to a limit which achieves the minimum of the problem (P1). Let $(\tilde{\phi}_1, ..., \tilde{\phi}_N, \tilde{\psi})$ be this limit. Then, by the Lagrange multiplier principle, this limit function $(\tilde{\phi}_1, ..., \tilde{\phi}_N, \tilde{\psi})$, after multiplying by a constant, corresponds to a solution of (1.3), at least in the distributional sense. Such solutions are called weak ground state solutions. But since the right sides of all $N+1$ equations in the system (1.3) are continuous functions, a standard bootstrapping argument shows that weak ground state solutions are indeed

classical solutions (see Proposition 2.13 below). In Section 2, we provide the details of the method.

In Section 3, we combine the variational formulation of solutions of (1.3) with the theory of symmetric decreasing rearrangements to prove the existence of solutions (Φ_j, Ψ) such that Φ_j and Ψ are even and decreasing positive functions in $(0, \infty)$ (see Theorems 3.1 and 3.5 below). Similar techniques have been used previously by Albert et al [3] to study solitary-wave solutions of some model equations for waves in stratified fluids, and by Angulo and Montenegro [4] to prove the existence and evenness of solitary waves for an interaction equation in two-layer fluid. The paper closes by showing in Theorem 3.6 the existence of a solitary wave for (1.1) with positive Fourier transforms. We state these results in the context of the existence theory for solitary waves introduced by Weinstein in [29] and will be proved by adapting an argument developed by Albert in [1].

Notation. For $(x, a) \in \mathbb{R} \times (0, \infty)$, we denote by $B(x, a)$ the ball centered at x and of radius a. In particular, we denote $B_a = B(0, a)$. The Fourier transform $\widehat{f}$ of a tempered distribution $f(x)$ on $\mathbb{R}$ is defined as $\widehat{f}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{i\xi x}\, dx$. If $1 \le r < \infty$, we shall denote by $L^r = L^r(\mathbb{R})$ the usual Banach space of Lebesgue measurable functions f on $\mathbb{R}$ for which the norm $|f|_{L^r}$ is finite, where

$$|f|_{L^r} = \left(\int_{-\infty}^{\infty} |f|^r \, dx \right)^{1/r} \quad \text{for} \ \ 1 \le r < \infty.$$

The space L^∞ consists of the measurable, essentially bounded functions on $\mathbb{R}$ with the norm $|f|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|$. The (Lebesgue) convolution of two functions f and g, denoted by $f \star g$, is the integral

$$f \star g(x) = f(x) \star g(x) = \int_{-\infty}^{\infty} f(x - \xi) g(\xi) \, d\xi.$$

For $s \ge 0$, we denote by $H^s = H^s(\mathbb{R})$ the Sobolev space of all tempered distributions f on $\mathbb{R}$ whose Fourier transforms $\widehat{f}$ are measurable functions on $\mathbb{R}$ satisfying

$$\|f\|_s^2 = \int_{-\infty}^{\infty} \left(1 + |\xi|^2 + ... + |\xi|^{2s}\right) |\widehat{f}(\xi)|^2 d\xi < \infty.$$

In particular, we use $\|f\|$ to denote the L^2 or H^0 norm of a function f. We define the space $\mathcal{Y}$ to be the cartesian product $H^1 \times ... \times H^1$ ($N+1$-times) provided with the product norm $\|\cdot\|_{\mathcal{Y}}$. For notational convenience, we denote

$$\begin{aligned} &(\mathbf{u}_j, v) = (u_1, ..., u_N, v), \\ &(\mathbf{u}_{j,n}, v_n) = (u_{1,n}, ..., u_{N,n}, v_n), \ \text{and} \ (\Phi_j, \Psi) = (\Phi_1, ..., \Phi_N, \Psi). \end{aligned}$$

In place of the compound subscripts, for example, when we take a subsequence of a sequence, we will follow the convention of using the same symbol to denote the subsequence. The letter C will be used to denote various positive constants which may assume different values from line to line but are not essential to the analysis of the

problem. The letter $C(...)$ will denote the constant whose value depends essentially only on the quantities indicated in the parentheses.

2. Existence of Solitary Waves

The main result of this section is the existence of global minimizers for the variational problem (P1):

Theorem 2.1. *Suppose that the assumptions* (1.4) *hold for the constants* c, σ, β, τ, γ, *and* α_j. *For* $\lambda > 0$, *define*

$$\mathcal{A} = \left\{ (f_1, ..., f_N, g) \in \mathcal{Y} : \int_{-\infty}^{\infty} F(f_1, ..., f_N, g)(x)\ dx = \lambda \right\}.$$

Then there exists a minimizing function for the problem (P1) in $\mathcal{A}$. *Consequently, the system* (1.3) *has a solution* $(\Phi_1, ..., \Phi_N, \Psi)$ *such that* $\Phi_1, ..., \Phi_N, \Psi$ *are in* $H^\infty(\mathbb{R})$ *and decay exponentially at infinity.*

In particular, Theorem 2.1 guarantees that the minimizing set $S(\lambda)$, namely

$$S(\lambda) = \{(\Phi_j, \Psi) \in \mathcal{A} : \mathsf{K}(\Phi_j, \Psi) = \inf \mathsf{K}(\mathbf{f}_j, g),\ (\mathbf{f}_j, g) \in \mathcal{A}\},$$

is non-empty. As will be seen below, this translates into an existence result for solitary-wave solutions (1.2) of (1.1).

We begin by proving some properties of the variational problem. The first lemma shows that K has a finite and positive infimum on $\mathcal{A}$.

Lemma 2.2. *For each* $\lambda > 0$, *one has*

$$I_\lambda = \inf \{\mathsf{K}(f_1, ..., f_N, g) : (f_1, ..., f_N, g) \in \mathcal{A}\} > 0. \tag{2.1}$$

Moreover, if $\lambda_2 > \lambda_1 > 0$, *then* $I_{\lambda_2} \geq I_{\lambda_1}$.

Proof. Denote $\Delta = (f_1, ..., f_N, g)$. From the Cauchy-Schwartz inequality and the Sobolev embedding theorem we have

$$\lambda = \int_{-\infty}^{\infty} F(\Delta)(x)\ dx \ \ \leq C \left(\|g\|_1 \|g\|^2 + \sum_{j=1}^{N} \|f_j\|_1 \|f_j\| \|g\| \right) \leq C \|\Delta\|_{\mathcal{Y}}^3, \tag{2.2}$$

where the constant C is independent of f_j, $1 \leq j \leq N$, and g. Now, using (2.2) it follows that

$$\begin{aligned} \mathsf{K}(\Delta) &\geq \min\{1, \sigma\} \sum_{j=1}^{N} \|f_j\|_1^2 + \min\{\gamma, c_\tau\} \|g\|_1^2 \\ &\geq \min\{\min\{1, \sigma\}, \min\{\gamma, c_\tau\}\} \|\Delta\|_{\mathcal{Y}}^2 \geq C \lambda^{2/3} > 0, \end{aligned}$$

and therefore $I_\lambda > 0$. To prove $I_{\lambda_2} \geq I_{\lambda_1}$, let $\epsilon > 0$ be arbitrary. There exists a function $\Theta = (\phi_1, ..., \phi_N, \psi)$ in $\mathcal{Y}$ such that $\int_{-\infty}^{\infty} F(\Theta)dx = \lambda_2$ and $\mathsf{K}(\Theta) < I_{\lambda_2} + \epsilon$. For $a \in \mathbb{R}$, denote

$$Q(a\Theta) = \int_{-\infty}^{\infty} F(\Theta(x))\ dx.$$

Then $Q(a\Theta)$ is a continuous function of $a \in \mathbb{R}$ and hence, using the intermediate value theorem of elementary analysis, we can find $\xi \in (0,1)$ such that $Q(\xi\Theta) = \lambda_1$. Hence

$$I_{\lambda_1} \leq \mathsf{K}(\xi\Theta) = \xi^2\mathsf{K}(\Theta) < \mathsf{K}(\Theta) < I_{\lambda_2} + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $I_{\lambda_1} \leq I_{\lambda_2}$, proving the lemma. □

By a minimizing sequence for I_λ in what follows, we mean to be any sequence $\{(\mathbf{f}_{j,n}, g_n)\}$ of functions in $\mathcal{A}$ satisfying the conditions

$$\lim_{n\to\infty} \mathsf{K}(\mathbf{f}_{j,n}, g_n) = I_\lambda \text{ and } \int_{-\infty}^{\infty} F(\mathbf{f}_{j,n}, g_n)(x)dx = \lambda,\ \forall n. \tag{2.3}$$

Lemma 2.3. *For all $\lambda > 0$ and $m > 1$, one has $I_{m\lambda} < mI_\lambda$.*

Proof. Let $\{(\mathbf{f}_{j,n}, g_n)\}$ be any sequence of functions in $\mathcal{A}$ satisfying (2.3). Denote $\Delta_n = (\mathbf{f}_{j,n}, g_n)$. Choose $\theta_n > 0$ such that

$$\int_{-\infty}^{\infty} F(\theta_n\Delta_n)(x)\ dx = m\lambda. \tag{2.4}$$

Since $\int_{-\infty}^{\infty} F(\Delta_n)dx = \lambda$, it follows from (2.4) that $\theta_n^3 = m > 1$. Thus

$$I_{m\lambda} \leq \mathsf{K}(\theta_n\Delta_n) = \frac{m}{\theta_n}\mathsf{K}(\Delta_n).$$

Since $m > 1$ and there exists $\epsilon > 0$ such that $\theta_n > 1 + \epsilon$ for sufficiently large n, the lemma follows by letting $n \to \infty$ in the last inequality. □

As an immediate corollary of Lemma 2.3, we obtain the following strict subadditivity property of I_λ :

Corollary 2.4. *Let I_λ be as defined in (2.1). Then, for all $\lambda_1, \lambda_2 > 0$,*

$$I_{(\lambda_1+\lambda_2)} < I_{\lambda_1} + I_{\lambda_1}.$$

Proof. Without loss of generality, we may assume that $\lambda_1 \geq \lambda_2$. If $\lambda_1 > \lambda_2$, then from what was shown in Lemma 2.3, it follows that

$$\begin{aligned} I_{(\lambda_1+\lambda_2)} &= I_{\lambda_1\left(1+\lambda_2\lambda_1^{-1}\right)} < \left(1 + \lambda_2\lambda_1^{-1}\right) I_{\lambda_1} \\ &\leq I_{\lambda_1} + \lambda_2\lambda_1^{-1}\left(\lambda_1\lambda_2^{-1} I_{\lambda_2}\right) = I_{\lambda_1} + I_{\lambda_2}; \end{aligned}$$

whereas in the case $\lambda_1 = \lambda_2$, we have

$$I_{(\lambda_1+\lambda_2)} = I_{2\lambda_1} < 2I_{\lambda_1} = I_{\lambda_1} + I_{\lambda_2},$$

so the corollary has been proved. □

We now proceed to prove the existence result. The idea is to show that any minimizing sequence $\{(\mathbf{f}_{j,n}, g_n)\}_{n\in\mathbb{N}}$ for I_λ in $\mathcal{A}$ which, up to subsequences and when its terms are suitably translated, has the following properties:

$$(\Phi_j, \Psi) = \lim_{n\to\infty} \Delta_n \in \mathcal{A} \quad \text{and } \mathsf{K}(\Delta_n) \le \liminf_{n\to\infty} \mathsf{K}(\Delta_n), \tag{2.5}$$

where $\Delta_n = (\mathbf{f}_{j,n}, g_n)$. Once we establish (2.5), the minimization problem (P1) is then solved since then it follows that

$$I_\lambda \le \mathsf{K}(\Phi_j, \Psi) \le \liminf_{n\to\infty} \mathsf{K}(\Delta_n) = I_\lambda,$$

where the first inequality holds because the limit pair (Φ_j, Ψ) belongs to $\mathcal{A}$ and the second inequality holds because $\{\Delta_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for I_λ. The key tool here is the concentration compactness principle developed by P. L. Lions [21], which has been used by many authors (see, for example, [2, 3, 4, 7, 8, 13, 32] and references therein). The method is based on the following lemma:

Lemma 2.5 (Lions [21])**.** *Let $\{Q_n\}_{n\ge 1}$ be a sequence of nonnegative functions in $L^1(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} Q_n(x)\ dx = \alpha$ for all n and some fixed $\alpha > 0$. Then there exists a subsequence $\{Q_{n_k}\}_{k\ge 1}$ satisfying exactly one of the following three possibilities:*

(1) *(Tightness up to translation) There are $y_k \in \mathbb{R}$ for $k = 1, 2, \ldots$, such that $Q_{n_k}(. + y_k)$ is tight, i.e., for any $\varepsilon > 0$, there is $R > 0$ large enough such that*

$$\int_{B(y_k,R)} Q_{n_k}(x)\ dx \ge \alpha - \epsilon \quad \textit{for all } k.$$

(2) *(Vanishing) For any $R > 0$,*

$$\lim_{k\longrightarrow\infty} \sup_{y\in\mathbb{R}} \int_{B(y,R)} Q_{n_k}(x)\ dx = 0.$$

(3) *(Dichotomy) There exists $\bar{\alpha} \in (0, \alpha)$ such that for any $\varepsilon > 0$, there exists $k_0 \ge 1$ and $Q_{1,k}, Q_{2,k} \in L^1_+(\mathbb{R})$ such that for $k \ge k_0$,*

$$\begin{cases} |Q_{n_k} - (Q_{1,k} + Q_{2,k})|_1 \le \epsilon, \quad \left|\int_{\mathbb{R}} Q_{1,k}\ dx - \bar{\alpha}\right| \le \epsilon, \\ \left|\int_{\mathbb{R}} Q_{2,k}\ dx - (\alpha - \bar{\alpha})\right| \le \varepsilon, \\ dist(supp(Q_{1,k}), supp(Q_{2,k})) \to \infty \quad as\ k \to \infty. \end{cases}$$

Remark 2.6. In Lemma 2.5 above, the condition $\int_{\mathbb{R}} Q_n(x)\ dx = \alpha$ can be replaced by $\int_{\mathbb{R}} Q_n(x)\ dx = \alpha_n$ where $\alpha_n \to \alpha > 0$ as $n \to \infty$. Indeed, it is enough to replace Q_n by Q_n/α_n and apply the lemma.

We now consider a minimizing sequence $\{(\mathbf{f}_{j,n}, g_n)\}_{n\in\mathbb{N}}$ for I_λ and apply the Lemma 2.5. Denote $\Delta_n = (\mathbf{f}_{j,n}, g_n)$ and let

$$Q_n = (g_n')^2 + g_n^2 + \sum_{j=1}^{N} \left((f_{j,n}')^2 + (f_{j,n})^2\right). \tag{2.6}$$

For each n, define $\mu_n = \int_{-\infty}^{\infty} Q_n(x)\ dx$. As $\{\Delta_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of real numbers is uniformly bounded for sufficiently large n. Without loss of generality, suppose that $\int_{-\infty}^{\infty} Q_n(x)\ dx \to \mu$ whenever $n \to \infty$. By Lemma 2.5 above, the sequence $\{Q_n\}_{n\in\mathbb{N}}$ has a subsequence which satisfies one of the three possibilities: Tightness up to translation, Vanishing, or Dichotomy. Our task is to show that tightness up to translation is the only possibility. To this end, suppose there is a subsequence $\{Q_{n_k}\}_{n\in\mathbb{N}}$ of $\{Q_n\}_{n\in\mathbb{N}}$ which satisfies either vanishing or dichotomy. We divide the proof into a sequence of lemmas. The first lemma rules out the vanishing condition:

Lemma 2.7. *Vanishing does not occur.*

Proof. We prove this lemma in several steps.
Step 1. Suppose $g \in C^\infty(\mathbb{R})$, and for $m \in \mathbb{Z}$, define $I_m = [m - 1/2, m + 1/2]$. Then for all $m \in \mathbb{Z}$, one has

$$\sup_{x\in I_m} |g(x)| \le \int_{I_m} |g(y)|\ dx + \int_{I_m} |g'(y)|\ dy. \tag{2.7}$$

To see this, for all $z \in I_m$ and $y \in I_m$, it is obvious that

$$g(z) = g(y) + \int_y^z g'(x)\ dx.$$

In consequence, one has for all $m \in \mathbb{Z}$,

$$|g(z)| \le |g(y)| + \int_{I_m} |g'(x)|\ dx.$$

Integrating both sides with respect to y over I_m, one obtains that

$$|g(z)| \le \int_{I_m} |g(y)|\ dy + \int_{I_m} |g'(y)|\ dy,$$

from which (2.7) follows.
Step 2. Suppose $\Delta = (f_1, \dots, f_n, g) \in \mathcal{Y}$ and $Q = Q_n$ be as defined in (2.6) with Δ_n replaced by the constant sequence $\Delta = \Delta_n$. Then there exists $C > 0$ such that

$$\int_{-\infty}^{\infty} |g|^3\ dx \le C \left(\sup_{y\in\mathbb{R}} \int_{B(y,1/2)} Q(x)\ dx\right)^{1/2} \|\Delta\|_{\mathcal{Y}}^2 \tag{2.8}$$

and for all $j = 1, \dots, N$,

$$\int_{-\infty}^{\infty} |g||f_j|^2\ dx \le C \left(\sup_{y\in\mathbb{R}} \int_{B(y,1/2)} Q(x)\ dx\right)^{1/2} \|\Delta\|_{\mathcal{Y}}^2. \tag{2.9}$$

To prove (2.8), assume first that $g \in C_0^\infty(\mathbb{R})$. By Step 1, replacing g by g^2, we obtain that

$$\left(\sup_{y\in I_m} |g(y)|\right)^2 \leq \|g^2(y)\|_{L^1(I_m)} + 2\int_{I_m} |g(y)||g'(y)|\ dy$$
$$\leq 2\|g(y)\|^2_{L^2(I_m)} + \|g'(y)\|^2_{L^2(I_m)} \leq C \sup_{y\in\mathbb{R}} \int_{B(y,1/2)} Q(x)\ dx.$$

Since $\int_{I_m} |g|^3\ dx \leq \|g\|_{L^\infty(I_m)} \|g\|^2_{L^2(I_m)}$, using the above estimate and taking the sum over all $m \in \mathbb{Z}$, it follows that

$$\int_{-\infty}^{\infty} |g|^3\ dx \leq C\left(\sup_{y\in\mathbb{R}} \int_{B(y,1/2)} Q(x)\ dx\right)^{1/2} \|\Delta\|^2_{\mathcal{Y}}.$$

This proves (2.8) for $g \in C_0^\infty(\mathbb{R})$. The result for $g \in H^1$ follows by approximating g with a sequence $\{g_n\} \subset C_0^\infty(\mathbb{R})$ such that $g_n \to g$ in H^1 norm. The proof for (2.9) uses the same argument.

Step 3. Suppose now that the vanishing case occurs, which is to say

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B(y,1/2)} Q_{n_k}(x)\ dx = 0.$$

But then using the estimates obtained in Step 2, it follows that

$$0 < \lambda = \left|\int_{-\infty}^{\infty} F(\Delta_{n_k})\ dx\right| \leq C\|\Delta_{n_k}\|^2_{\mathcal{Y}} \left(\sup_{y\in\mathbb{R}} \int_{B(y,1/2)} Q_{n_k}(x)\ dx\right)^{1/2} \to 0,$$

as $n \to \infty$, which is a contradiction. □

The next step in the proof of Theorem 2.1 is to rule out the dichotomy case. This is dealt with in the next three lemmas, which represent a simplification and generalization of arguments appeared in [2, 21].

Lemma 2.8. *Suppose there is a subsequence $\{Q_{n_k}\}_{k\in\mathbb{N}}$ of $\{Q_n\}_{n\in\mathbb{N}}$ such that the dichotomy alternative of Lemma 2.5 holds and denote $\Delta_{n_k} = (\mathbf{f}_{j,n_k}, g_{n_k})$. Then there exists a real number $\bar{\lambda} = \bar{\lambda}(\epsilon)$, a natural number n_0, and two sequences of functions $\{\Delta_k^{(1)}\}$ and $\{\Delta_k^{(2)}\}$ in Y satisfying $\Delta_{n_k} = \Delta_k^{(1)} + \Delta_k^{(2)}$ for all k and for all $k \geq k_0$,*

$$(i)\ \int_{-\infty}^{\infty} F(\Delta_k^{(1)})\ dx - \bar{\lambda} = O(\epsilon),$$
$$(ii)\ \int_{-\infty}^{\infty} F(\Delta_k^{(2)})\ dx - (\lambda - \bar{\lambda}) = O(\epsilon),$$
$$(iii)\ \mathsf{K}(\Delta_{n_k}) = \mathsf{K}(\Delta_k^{(1)}) + \mathsf{K}(\Delta_k^{(2)}) + O(\epsilon),$$

where the constants implied in the notation $O(\epsilon)$ can be chosen independently of n as well as ϵ. Furthermore, one has

$$\mathsf{K}(\Delta_k^{(1)}) \geq \bar{\mu} + O(\epsilon)\ \text{and}\ \mathsf{K}(\Delta_k^{(2)}) \geq \mu - \bar{\mu} + O(\epsilon), \tag{2.10}$$

where the real number $\bar{\mu}$ is as defined in Lemma 2.5.

Remark 2.9. The lemma says that the subsequence $\{\Delta_{n_k}\}_{k\in\mathbb{N}}$ can be split into two summands which carry fixed proportions of the constraint and which are supported far apart spatially that the sum of the values of the functional K at each summand does not exceed $\mathsf{K}(\Delta_{n_k})$.

Proof. If dichotomy case of Lemma 2.5 occurs, then there exists $\bar{\mu}\in(0,\mu)$ such that for any $\epsilon>0$ there corresponds $k_0\geq 1$ and L^1 functions $Q_{1,k}, Q_{2,k}\geq 0$ such that for all $k\geq k_0$,

$$\begin{aligned}&|Q_{n_k}-(Q_{1,k}+Q_{2,k})|_1\leq\varepsilon,\\ &\left|\int_{-\infty}^{\infty}Q_{1,k}\ dx-\bar{\mu}\right|\leq\varepsilon,\quad\text{and}\quad\left|\int_{-\infty}^{\infty}Q_{2,k}\ dx-(\mu-\bar{\mu})\right|\leq\varepsilon.\end{aligned}\tag{2.11}$$

Moreover, without loss of generality, we may assume that the supports of the functions $Q_{1,k}$ and $Q_{2,k}$ are separated as follows:

$$\begin{aligned}&\text{supp } Q_{1,k}\subset(y_k-R_0,y_k+R_0),\\ &\text{supp } Q_{2,k}\subset(-\infty,y_k-2R_k)\cup(y_k-2R_k,\infty),\end{aligned}\tag{2.12}$$

for some fixed $R_0>0$, a sequence of real numbers $\{y_k\}_{n\in\mathbb{N}}$, and $R_k\to\infty$. To split Δ_{n_k} into two summands $\Delta_k^{(1)}$ and $\Delta_k^{(2)},\ k=1,2,\ ...,$ let ζ and $\rho\in C_0^\infty(\mathbb{R})$ with $0\leq\zeta,\rho\leq 1$ be such that $\zeta\equiv 1$ on B_1, supp $\zeta\subset B_2$; $\rho\equiv 1$ on $\mathbb{R}\setminus B_2$, supp $\rho\subset\mathbb{R}\setminus B_1$. Denote the functions

$$\zeta_k(x)=\zeta\left(\frac{x-y_k}{R_1}\right),\quad\rho_k(x)=\rho\left(\frac{x-y_k}{R_k}\right),$$

where $x\in\mathbb{R}$, and $R_1>R_0$ chosen sufficiently large that

$$\left|\int_{-\infty}^{\infty}P(\zeta_k f_{j,n_k},\zeta_k g_{n_k})-Q_{1,k}\ dx\right|\leq\varepsilon\tag{2.13}$$

and

$$\left|\int_{-\infty}^{\infty}P(\rho_k f_{j,n_k},\rho_k g_{n_k})-Q_{2,k}\ dx\right|\leq\varepsilon.\tag{2.14}$$

In the last two inequalities we used the notation

$$P(\phi u_j,\phi v)=|(\phi v)'|^2+|\phi v|^2+\sum_{j=1}^{N}\left(|(\phi u_j)'|^2+|\phi u_j|^2\right).$$

To see that (2.13) and (2.14) are possible, first note that using the first inequality in (2.11) and the assumptions (2.12), we have that

$$\begin{aligned}&\int_{|x-y_k|\leq R_0}|Q_{n_k}-Q_{1,k}|\ dx\leq\varepsilon,\\ &\int_{|x-y_k|\geq 2R_k}|Q_{n_k}-Q_{2,k}|\,dx\leq\varepsilon,\ \int_{A(y_k;R_0,2R_k)}Q_{n_k}\ dx\leq\varepsilon.\end{aligned}\tag{2.15}$$

where $A(a;r,R)$ denotes the set $\{x : r \le |x-a| \le R\}$ for any $a \in \mathbb{R}, r > 0$, and $R > 0$. The left side of (2.13) can be written as

$$\begin{aligned} L &= \left| \int_{|x-y_k|\le 2R_1} P(\zeta_k f_{j,n_k}, \zeta_k g_{n_k}) - Q_{1,k}\ dx \right| \\ &= \left| \int_{|x-y_k|\le R_0} Q_{n_k} - Q_{1,k}\ dx \right| + \max_{x\in\mathbb{R}} \Omega(R_1;\zeta(x)) \int_{A(y_k;R_0,2R_1)} Q_{n_k}\ dx, \end{aligned}$$

where for any $a \in \mathbb{R}$ and $\varphi \in C_0^\infty$, $\Omega(a;\varphi(x))$ is given by

$$\Omega(a;\varphi(x)) = |\varphi(x)|^2 + \frac{1}{a}|\varphi'(x)|^2,\ x \in \mathbb{R}.$$

Using relations (2.15), we have that $L \le \epsilon + \epsilon = O(\epsilon)$, as $\epsilon \to 0$. Similarly, we write the left side of the inequality (2.14) as

$$\begin{aligned} L_1 &= \left| \int_{|x-y_k|\ge R_k} P(\rho_k f_{j,n_k}, \rho_k g_{n_k}) - Q_{2,k}\ dx \right| \\ &\le \left| \int_{A(y_k;R_k,2R_k)} P(\rho_k f_{j,n_k}, \rho_k g_{n_k}) - Q_{2,k}\ dx \right| + \left| \int_{|x-y_k|\ge 2R_k} Q_{n_k} - Q_{2,k}\ dx \right| \\ &\le \max_{x\in\mathbb{R}} \Omega(R_k;\rho(x)) \int_{A(y_k;R_k,2R_k)} Q_{n_k}\ dx + \int_{|x-y_k|\ge 2R_k} |Q_{n_k} - Q_{2,k}|\ dx, \end{aligned}$$

and hence, $L_1 \le \epsilon + \epsilon = O(\epsilon)$, as $\epsilon \to 0$. Let us now define $\Delta_k^{(1)}$ and $\Delta_k^{(2)}$ by setting

$$\begin{cases} \Delta_k^{(1)} = (\mathbf{f}_{j,k}^{(1)}, g_k^{(1)}) = (\zeta_k \mathbf{f}_{j,n_k}, \zeta_k g_{n_k}) = \zeta_k \Delta_{n_k}, \\ \Delta_k^{(2)} = (\mathbf{f}_{j,k}^{(2)}, g_k^{(2)}) = (\rho_k \mathbf{f}_{j,n_k}, \rho_k g_{n_k}) = \rho_k \Delta_{n_k}, \end{cases}$$

and let $\Theta_k = (\mathbf{u}_{j,k}, v_k)$ be such that $\Delta_{n_k} = \Delta_k^{(1)} + \Delta_k^{(2)} + \Theta_k$. Then $\Delta_k^{(1)}, \Delta_k^{(2)}, \Theta_k$ are all in $\mathcal{Y}$. Since $\int_{-\infty}^{\infty} |F(\Delta_k^{(1)})|\ dx$ is bounded, there exists a subsequence of $\{\Delta_k^{(1)}\}_{k\in\mathbb{N}}$, which we denote again by the same symbol, and a positive real number $\bar\lambda = \bar\lambda(\epsilon)$ such that $\int_{-\infty}^{\infty} F(\Delta_k^{(1)})\ dx \to \bar\lambda$. Then, for sufficiently large k,

$$\int_{-\infty}^{\infty} F(\Delta_k^{(1)})\ dx - \bar\lambda = O(\epsilon). \tag{2.16}$$

To estimate the proportion of the constraint functional carried by $\Delta_k^{(2)}$, we write the integral $\int_{-\infty}^{\infty} F(\Delta_{n_k})\ dx$ as

$$\int_{-\infty}^{\infty} F(\Delta_k^{(1)}) + \int_{-\infty}^{\infty} F(\Delta_k^{(2)}) + \int_{A(y_k;R_0,2R_k)} \left[F(\Delta_{n_k}) - F(\Delta_k^{(1)}) - F(\Delta_k^{(1)}) \right], \tag{2.17}$$

where all integrals are with respect to x. The last integral in this equation is estimated as follows:

$$\begin{aligned}\int_{A(y_k;R_0,2R_k)} \left[F(\Delta_{n_k}) - F(\Delta_k^{(1)}) - F(\Delta_k^{(1)})\right]\ dx &\le C\|\Theta_k\|_{\mathcal{Y}}^2 \\ \le \max\left\{|1-\zeta_k-\eta_k|_\infty^2, \frac{|\zeta'|_\infty^2}{R_1^2} + \frac{|\eta'|_\infty^2}{R_k^2}\right\} \int_{A(y_k;R_1,2R_k)} Q_{n_k}\ dx = O(\epsilon),\end{aligned}$$

as $\epsilon \to 0$. Thus, from (2.17), we can conclude that

$$\int_{-\infty}^{\infty} F(\Delta_{n_k})\ dx = \int_{-\infty}^{\infty} F(\Delta_k^{(1)})\ dx + \int_{-\infty}^{\infty} F(\Delta_k^{(2)})\ dx + O(\epsilon).$$

It then follows by taking the limit of both sides as $k \to \infty$ that

$$\int_{-\infty}^{\infty} F(\Delta_k^{(2)})\ dx = \lambda - \bar{\lambda} + O(\epsilon).$$

To prove the assertion that the sum of the values of K at $\Delta_k^{(1)}$ and $\Delta_k^{(2)}$ does not exceed $\mathsf{K}(\Delta_{n_k})$, we write

$$\begin{aligned}\mathsf{K}(\Delta_{n_k}) &= \mathsf{K}\left(f_{1,k}^{(1)} + f_{1,k}^{(2)} + u_{1,k},\ \ldots, f_{N,k}^{(1)} + f_{N,k}^{(2)} + u_{N,k}, g_k^{(1)} + g_k^{(2)} + v_k\right) \\ &= \mathsf{K}(\mathbf{f}_{j,k}^{(1)}, g_k^{(1)}) + \mathsf{K}(\mathbf{f}_{j,k}^{(2)}, g_k^{(2)}) + \mathsf{K}(\mathbf{u}_{j,k}, v_k) + \sum_{j=1}^{N} J_j + J,\end{aligned} \tag{2.18}$$

where the integrals J and J_j on the right-hand side are given by

$$\begin{aligned}J = \gamma \int_{-\infty}^{\infty} &\left[(v_k)'(\zeta_k g_{n_k})' + (v_k + (\zeta_k g_{n_k})')\,(\rho_k g_{n_k})'\right]\ dx \\ &+ c_\tau \int_{-\infty}^{\infty} \left[v_k \zeta_k g_{n_k} + (v_k + \zeta_k g_{n_k})\rho_k g_{n_k}\right]\ dx\end{aligned}$$

and for each $j = 1, ..., N$,

$$\begin{aligned}J_j = \int_{-\infty}^{\infty} &\left[(u_{j,k})'(\zeta_k f_{j,n_k})' + ((u_{j,k})' + (\zeta_k f_{j,n_k})')\,(\rho_k f_{j,n_k})'\right]\ dx \\ &+ \sigma \int_{-\infty}^{\infty} \left[u_{j,k}\zeta_k f_{j,n_k} + (u_{j,k} + \zeta_k f_{j,n_k})\,\rho_k f_{j,n_k}\right]\ dx.\end{aligned}$$

From the Cauchy-Schwarz inequality, it follows that

$$J \le C\|\Theta_k\|_{\mathcal{Y}} \cdot \|g_{n_k}\| = O(\epsilon) \quad \text{and} \quad J_j \le C\|\Theta_k\|_{\mathcal{Y}} \cdot \|f_{j,n_k}\| = O(\epsilon),$$

where $C = C(\zeta_k, \rho_k)$. Thus, from (2.18), we obtain that

$$\mathsf{K}(\Delta_{n_k}) = \mathsf{K}(\Delta_k^{(1)}) + \mathsf{K}(\Delta_k^{(2)}) + O(\epsilon).$$

To complete the proof of Lemma 2.8, it only remains to establish inequalities in (2.10). To establish the first inequality, we see that

$$\begin{aligned}
C\|\Delta_k^{(1)}\|_{\mathcal{Y}}^2 &\geq \mathsf{K}(\Delta_k^{(1)}) = \mathsf{K}(\zeta_k f_{1,n_k}, ..., \zeta_k f_{N,n_k}, \zeta_k g_{n_k}) \\
&= O(\epsilon) + \int_{-\infty}^{\infty} \zeta_k^2 (f_{1,n_k}, \dots, f_{N,n_k}, g_{n_k}) D(L_{ii}) (f_{1,n_k}, \dots, f_{N,n_k}, g_{n_k})^T \, dx \\
&\geq O(\epsilon) + \int_{-\infty}^{\infty} \zeta_k^2 Q_{n_k} \, dx = O(\epsilon) + \int_{B(y_k,R_0)} Q_{n_k} \, dx + \int_{A(y_k;R_0,2R_k)} \zeta_k^2 Q_{n_k} \, dx \\
&= \int_{-\infty}^{\infty} Q_{1,k} \, dx + O(\epsilon) \geq \bar{\mu} + O(\epsilon),
\end{aligned}$$

so the first inequality in (2.10) has been proved. The second inequality in (2.10) can be proved similarly. $\square$

Lemma 2.10. *Let Q_n be as defined in* (2.6) *and that the dichotomy case occurs. Then there exists $\theta \in (0, \lambda)$ such that*

$$I_\lambda \geq I_\theta + I_{(\lambda-\theta)}.$$

Proof. Let $\bar{\lambda} = \bar{\lambda}(\epsilon)$ be as defined in Lemma 2.8. Since $\int_{-\infty}^{\infty} F(\Delta_k^{(1)})dx$ is bounded, the range of values of $\bar{\lambda}(\epsilon)$ remains bounded as $\epsilon \to 0$. Thus, by restricting attention to a sequence of values of ϵ tending to 0 and extracting an appropriate subsequence from this sequence, we may assume that $\bar{\lambda}(\epsilon) \to \theta$ as $\epsilon \to 0$. It is claimed that $\theta \in (0, \lambda)$. To see this, first notice that from

$$\mathsf{K}(\Delta_{n_k}) = \mathsf{K}(\Delta_k^{(1)}) + \mathsf{K}(\Delta_k^{(2)}) + O(\epsilon),$$

it follows immediately that

$$I_\lambda = \liminf_k \mathsf{K}(\Delta_{n_k}) \geq \liminf_k \mathsf{K}(\Delta_k^{(1)}) + \liminf_k \mathsf{K}(\Delta_k^{(2)}) + O(\epsilon). \tag{2.19}$$

Suppose for the sake of contradiction that $\theta \leq 0$. Then we have that

$$\int_{-\infty}^{\infty} F(\Delta_k^{(2)})(x) \, dx = \lambda - \theta + O(\epsilon)$$

for sufficiently large n. Let us now define $\bar{\Delta}_k^{(2)} = \phi_k \Delta_k^{(2)}$, where ϕ_k is chosen such that $\int_{-\infty}^{\infty} F(\bar{\Delta}_k^{(2)})dx = \lambda - \theta$. Then $\phi_k = 1 + O(\epsilon)$ and

$$\mathsf{K}(\Delta_k^{(2)}) = \frac{1}{\phi_k^2} \mathsf{K}(\bar{\Delta}_k^{(2)}) \geq \frac{1}{\phi_k^2} I_{\lambda-\theta} \geq \frac{1}{(1+O(\epsilon))^2} I_\lambda, \tag{2.20}$$

where the last inequality is a consequence of Lemma 2.2. From (2.19), (2.20), and the first inequality of (2.10), it follows that

$$I_\lambda \geq C\bar{\mu} + \frac{1}{(1+O(\epsilon))^2} I_\lambda + O(\epsilon).$$

As $\epsilon \to 0$, the last inequality gives $I_\lambda \geq C\bar{\mu} + I_\lambda > I_\lambda$, a contradiction.

On the other hand, if it were the case that $\theta \geq \lambda$, then we would have $\int_{-\infty}^{\infty} F(\Delta_k^{(1)})\, dx = \theta + O(\epsilon)$ for sufficiently large n, and a similar argument as in the case $\theta \leq 0$ would show that (2.19) yields

$$I_\lambda \geq C(\mu - \bar{\mu}) + \frac{1}{(1+O(\epsilon))^2} I_\lambda + O(\epsilon),$$

which implies $I_\lambda \geq C(\mu - \bar{\mu}) + I_\lambda > I_\lambda$, another contradiction. This proves the claim that $\theta \in (0, \lambda)$.

Finally, as a consequence the above arguments, one also obtains that

$$I_\lambda \geq \frac{1}{(1+O(\epsilon))^2} I_\theta + \frac{1}{(1+O(\epsilon))^2} I_{\lambda-\theta} + O(\epsilon),$$

which upon taking limit as $\epsilon \to 0$, gives $I_\lambda \geq I_\theta + I_{\lambda-\theta}$, proving the lemma. □

We can now rule out the dichotomy condition:

Lemma 2.11. *The dichotomy does not occur.*

Proof. This follows from Lemma 2.10 and Corollary 2.4. □

With both vanishing and dichotomy alternatives ruled out, we can now complete the proof of Theorem 2.1. Because vanishing and dichotomy have been ruled out, Lemma 2.5 guarantees that sequence $\{Q_n\}$ is tight, i.e., there exists a sequence of real numbers $\{y_n\}_{n\in\mathbb{N}}$ such that for any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ so that for all $n \in \mathbb{N}$,

$$\int_{|x-y_n|\leq R} Q_n(x)\ dx \geq \mu - \varepsilon, \quad \int_{|x-y_n|\geq R} Q_n(x)\ dx \leq \varepsilon,$$

and

$$\left| \int_{|x-y_n|\geq R} F\left(\mathbf{f}_{j,n}, g_n\right)\ dx \right| \leq C\ \|(\mathbf{f}_{j,n}, g_n)\|_{\mathcal{Y}} \int_{|x-y_n|\geq R} Q_n(x)\ dx = O(\epsilon),$$

as $\epsilon \to 0$. It then follows that for n large enough,

$$\left| \int_{|x-y_n|\leq R} F\left(\mathbf{f}_{j,n}, g_n\right)\ dx - \lambda \right| \leq \epsilon. \tag{2.21}$$

Denote by $w_{j,n}, 1 \leq j \leq N$, and z_n the translated functions

$$w_{j,n}(x) = f_{j,n}(\cdot + y_n), \quad z_n(x) = g_n(\cdot + y_n).$$

Then, $\{(\mathbf{w}_{j,n}, z_n)\}$ is bounded in $\mathcal{Y}$, and hence by the Banach-Alaoglu theorem, there exists a subsequence, we again label by $\{(\mathbf{w}_{j,n}, z_n)\}$, which converges weakly in $\mathcal{Y}$ to a vector-function $(\Phi_1, ..., \Phi_N, \Psi)$. It then follows immediately from (2.21) that for $n \geq n_0$,

$$\lambda \geq \int_{-R}^{R} F\left(w_{1,n}, ..., w_{N,n}, z_n\right)\ dx \geq \lambda - \epsilon. \tag{2.22}$$

Since $H^1([-R,R])$ is compactly embedded in $L^2([-R,R])$, we have

$$\begin{aligned}\int_{-R}^{R} \left|w_{1,n}^2 z_n - \Phi_1^2\Psi\right| \, dx &\le |w_{1,n}+\Phi_1|_\infty \cdot \|z_n\| \cdot \|w_{1,n}-\Phi_1\|_{L^2(-R,R)} \\ &\quad + \|w_{1,n}\|_1^2 \cdot \|z_n-\Psi\|_{L^2(-R,R)} \\ &\le C\left(\|w_{1,n}-\Phi_1\|_{L^2(-R,R)} + \|z_n-\Psi\|_{L^2(-R,R)}\right) \to 0,\end{aligned}$$

as $n\to\infty$. Similarly, $\int_{-R}^{R} w_{j,n}^2 z_n \, dx \to \int_{-R}^{R} \Phi_j^2\Psi \, dx$ for all $2\le j\le N$. We also have

$$|z_n-\Psi|_{L^3(-R,R)} \le C\|z_n-\Psi\|_1^{1/6}\|z_n-\Psi\|_{L^2(-R,R)}^{5/6} \le C\|z_n-\Psi\|_{L^2(-R,R)}^{5/6},$$

and hence, $\int_{-R}^{R} z_n^3 \, dx \to \int_{-R}^{R} \Psi^3 \, dx$. Therefore, from (2.22), we have that

$$\lambda \ge \int_{-R}^{R} F\left(\Phi_1,...,\Phi_N,\Psi\right) \, dx \ge \lambda - \epsilon.$$

Thus, for $\epsilon = 1/j,\ j\in\mathbb{N}$, there exists $R_j > j$ such that

$$\lambda \ge \int_{-R_j}^{R_j} F\left(\Phi_1,...,\Phi_2,\Psi\right) \, dx \ge \lambda - \frac{1}{j},$$

and consequently, as $j\to\infty$, we have that $(\Phi_j,\Psi)\in\mathcal{A}$. Furthermore, from the weak lower semicontinuity of K and the invariance K by translations, we have

$$I_\lambda = \lim_{n\to\infty} \mathsf{K}(\mathbf{f}_{j,n}, g_n) \ge \mathsf{K}(\Phi_j,\Psi) \ge I_\lambda,$$

and thus, (Φ_j,Ψ) must be a minimizer for I_λ, i.e., $(\Phi_j,\Psi)\in S(\lambda)$. But then (Φ_j,Ψ) must satisfy the Euler-Lagrange equation for (P1), i.e., there exists some multiplier $\kappa\in\mathbb{R}$ (Lagrange multiplier) such that

$$\begin{cases} -\Phi_1'' + \sigma\Phi_1 = \kappa\alpha_1\Psi\Phi_1, \\ \quad \cdots \qquad \cdots \qquad \cdots \\ -\Phi_N'' + \sigma\Phi_N = \kappa\alpha_N\Psi\Phi_N, \\ -\gamma\Psi'' + c_\tau\Psi = \dfrac{\kappa}{2}\left(\beta\Psi^2 + \alpha_1\Phi_1^2 + \ldots + \alpha_1\Phi_N^2\right). \end{cases} \tag{2.23}$$

An easy calculation proves that the Lagrange multiplier is positive:

Proposition 2.12. *The Lagrange multiplier satisfies $\kappa > 0$.*

Proof. Multiplying the first and second equations above by Φ_j and Ψ, respectively, and integrating over the real line, we obtain

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} \left((\Phi_j')^2 + \sigma\Phi_j^2\right) \, dx = k\int_{-\infty}^{\infty} \alpha_j \ \Psi\Phi_j^2 \, dx, \ j=1,2,...,N, \\ \displaystyle\int_{-\infty}^{\infty} \left(\gamma(\Psi')^2 + c_\tau\Psi^2\right) \, dx = \frac{\kappa}{2}\int_{-\infty}^{\infty} \left(\beta \ \Psi^3 + \sum_{j=1}^{N}\alpha_j \ \Phi_j^2\Psi\right) \, dx. \end{cases}$$

Adding these $N+1$ equations and using the facts that $\mathsf{K}(\Phi_j, \Psi) = I_\lambda$ and $\int_{-\infty}^{\infty} F(\Phi_j, \Psi)\, dx = \lambda$, we obtain

$$\kappa = \frac{2}{3\lambda} I_\lambda > 0,$$

which is the desired result. □

Finally, we see that these equations (2.23) are satisfied by $\Phi_j, 1 \le j \le N$, and Ψ if and only if the functions u_j and v defined by

$$u_j(x,t) = \kappa e^{i\omega t} e^{ic(x-ct)/2} \Phi_j(x-ct),\ v(x,t) = \kappa \Psi(x-ct)$$

are solutions of (1.1). That is, solutions to the variational problem (P1) corresponds to solitary-wave profiles of (1.1).

To complete the proof of Theorem 2.1, it only remains to prove smoothness and exponential decay of the solutions:

Proposition 2.13. *Suppose* $(\Phi_1, ..., \Phi_N, \Psi) \in \mathcal{Y}$ *is a solution of* (1.3)*, in the sense of distributions. Then*

(i) $\Phi_1, ..., \Phi_N, \Psi \in H^\infty(\mathbb{R})$.
(ii) *One has pointwise exponential decay, i.e.,*

$$|\Phi_j(x)| \le Ce^{-\delta_j|x|},\ \textit{and}\ |\Psi(x)| \le Ce^{-\delta|x|},$$

holds for all $x \in \mathbb{R}$*, where* $\delta, \delta_j > 0$*, and* $C > 0$ *are suitable constants.*

Proof. Statement (i) follows by a standard bootstrap argument. Since $\Phi_1, ..., \Phi_N$, Ψ are H^1 functions, and H^1 is an algebra, it follows that Ψ^2, Φ_j^2, and $\Phi_j\Psi, 1 \le j \le N$, are also H^1 functions. Since the convolution operation with K_s takes H^s to H^{s+2} for any $s \ge 0$, so (3.1) implies that $\Phi_j, ..., \Phi_N, \Psi$ are in H^3. But then Ψ^2, Φ_j^2, and $\Phi_j\Psi$ are H^3 functions, so (3.1) implies that Φ_j and Ψ are in H^5, and so on. Continuing this argument inductively proves that $\Phi_1, ..., \Phi_N$, Ψ are in H^∞.

To prove decay estimates, we borrow an argument from the proof of Theorem 8.1.1 of [10]. Fix $j \in \{1, 2, ..., N\}$. For $\epsilon > 0$ and $\delta > 0$, consider the function $\varphi(x) = e^{\epsilon|x|/(1+\delta|x|)} \in L^\infty(\mathbb{R})$. Multiplying the first equation in (1.3) by $\varphi\Phi_j$, we get

$$-\int_{-\infty}^{\infty} \varphi\ \Phi_j''\Phi_j\ dx + \sigma \int_{-\infty}^{\infty} \varphi\ \Phi_j^2\ dx = \alpha_j \int_{-\infty}^{\infty} \varphi\ \Phi_j^2\Psi\ dx.$$

Integrating by parts and using the fact that $\varphi' \le \epsilon\varphi$, we get

$$\sigma \int_{-\infty}^{\infty} \varphi\Phi_j^2\ dx \le \int_{-\infty}^{\infty} \varphi(\Phi_j')^2\ dx + \epsilon \int_{-\infty}^{\infty} \varphi|\Phi_j\Phi_j'|\ dx + \alpha_j \int_{-\infty}^{\infty} \varphi\Phi_j^2|\Psi|\ dx.$$

Now using the Cauchy-Schwarz inequality, we obtain from the preceding inequality that

$$\begin{aligned}\left(\sigma - \frac{\epsilon}{2}\right) \int_{-\infty}^{\infty} \varphi\Phi_j^2\ dx &\le \left(1 + \frac{\epsilon}{2}\right) \int_{-\infty}^{\infty} \varphi(\Phi_j')^2\ dx + \alpha_j \int_{-\infty}^{\infty} \varphi\Phi_j^2|\Psi|\ dx \\ &\le \alpha_j \int_{-\infty}^{\infty} \varphi\Phi_j^2|\Psi|\ dx,\end{aligned}$$

with ϵ chosen to be sufficiently small. Thus, for ϵ small enough, we deduce that

$$\int_{-\infty}^{\infty} \varphi(x)\Phi_j^2(x)\ dx \leq C \int_{-\infty}^{\infty} \varphi(x)\Phi_j^2(x)|\Psi(x)|\ dx, \tag{2.24}$$

where $C = C(\epsilon, \sigma, \alpha_j)$ (independent of δ). Since Ψ is an H^1 function, then $\Psi(x) \to 0$ as $|x| \to \infty$. We can find $R > 0$ such that $|\Psi(x)| \leq 1/(2C)$ for $|x| \geq R$. It then follows from (2.24) that

$$\int_{-\infty}^{\infty} \varphi(x)\Phi_j^2(x)\ dx \leq 2C \int_{B_R} \varphi(x)\Phi_j^2(x)|\Psi(x)|\ dx.$$

Taking $\delta \to 0$, Fatou's lemma and Lebesgue's theorem yields

$$\int_{-\infty}^{\infty} e^{\epsilon|x|}|\Phi_j(x)|^2\ dx < \infty,\ \ j = 1, 2, ..., N. \tag{2.25}$$

Now since Φ_j belongs to H^1, then $\Phi_j(x) \to 0$ as $|x| \to \infty$ and Φ_j is globally Lipschitz continuous on $\mathbb{R}$. From these two properties of Φ_j and (2.25), one can easily show that $e^{\delta_1|x|}\Phi_j(x)$ is bounded on $\mathbb{R}$ for some $0 < \delta_1 \leq \epsilon$ (for details, the reader may consult the proof of Theorem 8.1.1 in [10]).

To obtain the decay estimate for Ψ, multiplying the second equation in (1.3) by $\varphi\Psi$ gives, as above, the following estimate

$$\int_{-\infty}^{\infty} \varphi\ \Psi^2\ dx \leq C \int_{-\infty}^{\infty} \varphi\ \left(|\Psi|^3 + \sum_{j=1}^{N} \Phi_j^2|\Psi|\right)\ dx,$$

where $C = C(\epsilon, \gamma, c, \tau, \beta, \alpha_j)$. Choose $\epsilon < 2\delta_1$. Then, using decay estimates for the functions $\Phi_1, ..., \Phi_N$ proved above, we can show, as before, that $\int_{-\infty}^{\infty} \varphi\Psi^2\ dx$ is bounded by some constant which is independent of δ. Then, taking $\delta \to 0$. allows us to deduce that

$$\int_{-\infty}^{\infty} e^{\epsilon|x|}|\Psi(x)|^2\ dx < \infty,$$

and from here we can proceed as we did for $\Phi_j(x)$. □

The proof of Theorem 2.1 is now complete.

3. **Properties of Solitary Waves**

In this section we establish some properties of travelling solitary waves. To do so, we take advantage of the convolution representation for solutions of the equation (1.3). We assume throughout this section, unless otherwise stated, that the assumptions (1.4) hold with $\gamma > 0$.

Provided $\eta = c_\tau/\gamma > 0$ in (1.3), we can rewrite (1.3) in the form

$$\begin{cases} \Phi_j = \alpha_j \ K_\sigma \star \Phi_j \Psi, \quad 1 \le j \le N, \\ \Psi = \dfrac{1}{2\gamma} \ K_\eta \star \left(\beta \Psi^2 + \displaystyle\sum_{j=1}^{N} \alpha_j \Phi_j^2 \right), \end{cases} \tag{3.1}$$

where for any $s > 0$, the kernel K_s is defined explicitly in terms of its Fourier symbol

$$q(\xi) = \widehat{K}_s(\xi) = \frac{1}{s + \xi^2}. \tag{3.2}$$

The kernel K_s defined via its Fourier transform as in (3.2) is a real-valued, even, bounded, continuous function, and $K_s(x) \to 0$ as $|x| \to \infty$. Furthermore, K_s is strictly positive on $\mathbb{R}$. To see this, one can use the Residue Theorem and Jordan's lemma (see, for example, Chapter 3 of [27]) to represent K_s explicitly in the form

$$K_s(x) = \frac{1}{2\sqrt{s}} \sqrt{\frac{\pi}{2}} e^{-\sqrt{s}|x|}, \ s > 0, \ x \in \mathbb{R}. \tag{3.3}$$

The first property concerns signs of $\Phi_j, 1 \le j \le N$, and Ψ :

Theorem 3.1. *Every solution* $(\Phi_1, ..., \Phi_N, \Psi)$ *of the* (1.3) *satisfies the following properties:*

(i) $\Psi(x) > 0$ *for all* $x \in \mathbb{R}$.

(ii) *The functions* $\Phi_1(x), ..., \Phi_N(x)$ *are of one sign on* $\mathbb{R}$.

Proof. To prove (i), we use the convolution representation of Ψ, namely

$$\Psi(x) = \frac{1}{2\gamma} \int_{-\infty}^{\infty} K_\eta(x - y) Q(y) \ dy \text{ with } Q(y) = \beta \Psi^2(y) + \sum_{j=1}^{N} \alpha_j \ \Phi_j^2(y). \tag{3.4}$$

Since the kernel K_η is a strictly positive and $Q(x)$ is everywhere non-negative, it then follows from (3.4) that $\Psi(x) > 0$ everywhere provided that $(\Phi_1, ..., \Phi_N, \Psi)$ is a solution of (1.3) with at least one component being nonzero on a set of positive measure.

To prove (ii), denote $\Theta = (\Phi_1, ..., \Phi_N, \Psi)$. Let $U_j = |\Phi_j|, j = 1, .., N$, and denote $\Delta = (U_1, ..., U_N, \Psi)$. It is a standard fact from real analysis that if $\Phi_j \in H^1$, then $|\Phi_j(x)|$ is in H^1 and

$$\int_{-\infty}^{\infty} ||\Phi_j|_x|^2 \ dx \le \int_{-\infty}^{\infty} |(\Phi_j)_x|^2 \ dx. \tag{3.5}$$

(For a proof of this elementary fact, readers may consult Lemma 3.4 of [3].) Then $\Delta \in \mathcal{Y}$ and using (3.5), it follows that $\mathsf{K}(\Delta) \le \mathsf{K}(\Theta)$. Since $\int_{-\infty}^{\infty} F(\Delta) \ dx = \int_{-\infty}^{\infty} F(\Theta) \ dx$, thus Δ and Θ are both in $S(\lambda)$ for some $\lambda > 0$. Observe that since Θ and Δ satisfy the same equations (2.23), we have that for each $j = 1, ..., N$,

$$\begin{cases} -\Phi_j''(x) + \sigma \Phi_j(x) = \kappa \alpha_j \ \Psi(x) \Phi_j(x), \\ -U_j''(x) + \sigma U_j(x) = \kappa \alpha_j \ \Psi(x) U_j(x). \end{cases} \tag{3.6}$$

Multiplying the first equation in (3.6) by U_j and the second by Φ_j and subtracting the second from the first, we see that the Wronskian

$$W(\Phi_j(x), U_j(x)) = \begin{vmatrix} \Phi_j(x) & U_j(x) \\ \Phi_j'(x) & U_j'(x) \end{vmatrix} = \text{constant}.$$

But since $W(\Phi_j, U_j) \to 0$ as $x \to \infty$, we must have $W(\Phi_j, U_j) = 0$ for all $x \in \mathbb{R}$. Then the functions Φ_j and $U_j, 1 \leq j \leq N$, are linearly dependent and so, $\Phi_j, 1 \leq j \leq N$, must be of one sign on $\mathbb{R}$. This completes the proof. □

We now use the theory of symmetric decreasing rearrangement to prove the existence of a solution $(\Phi_1, ..., \Phi_N, \Psi)$ of (1.3) such that $\Phi_1, ..., \Phi_N, \Psi$ are even, strictly positive, and decreasing functions on $(0, \infty)$. Recall that, for a non-negative function $w : \mathbb{R} \to [0, \infty)$, one may define the symmetric rearrangement of w to be the unique function w^* with domain $\mathbb{R}$ which has the same distribution function as w, that is, for every $a > 0$, the sets $\{x : |w(x)| > a\}$ and $\{x : |w^*(x)| > a\}$ have the same measure. In formulas,

$$w^*(x) = \inf \left\{ a > 0 : \frac{1}{2} m(w, a) \leq |x| \right\} = \sup \left\{ a > 0 : \frac{1}{2} m(w, a) > |x| \right\},$$

where $m(w, a)$ denotes the measure of $\{x : |w(x)| > a\}$ for all $a > 0$ (or see Chapter 2 of [20] for a slightly different but equivalent definition and also a comprehensive discussion of many different types of rearrangements). The function w^* is clearly radially symmetric and non-increasing in the variable $|x|$, i.e., $w^*(x) = w^*(y)$ if $|x| = |y|$ and $w^*(x) \geq w^*(y)$ if $|x| \leq |y|$.

The next two theorems about symmetric decreasing rearrangements play a crucial role in the rest of the paper.

Theorem 3.2. *The following statements hold:*

(i) *Rearrangement preserves L^p norms, i.e., for every nonnegative function f in L^p,*

$$|f|_p = |f^*|_p, \ 1 \leq p \leq \infty.$$

(ii) *(Hardy-Littlewood Inequality) If f and g are nonnegative measurable functions that vanish at infinity, then*

$$\int_{-\infty}^{\infty} f(x)g(x) \ dx \leq \int_{-\infty}^{\infty} f^*(x)g^*(x) \ dx.$$

(iii) *(Pólya-Szegő Inequality) The symmetric decreasing rearrangement diminishes L^2 norm of the gradient of a positive function f in H^1 :*

$$\int_{-\infty}^{\infty} |(u^*)_x|^2 \ dx \leq \int_{-\infty}^{\infty} |u_x|^2 \ dx.$$

For proofs of these statements, as well as other basic facts about rearrangements, reader may consult, for example, the appendix of [4].

Theorem 3.3 (F. Riesz)**.** *Let $f_1, ..., f_N$ be measurable functions on $\mathbb{R}$ such that $m\{x : f_j \geq y\} < \infty$ for all $y > 0$ and all $1 \leq j \leq N$. Then*

$$|(f_1 \star f_2 \star ... \star f_N)(0)| \leq [(f_1^*) \star (f_2^*) \star ... \star (f_N^*)]\,(0) \tag{3.7}$$

in the sense that if the right-hand side is finite, then the left-hand side exists and the inequality holds.

A proof of Theorem 3.3 for $N = 3$, along with a sketch of the inductive proof for $N \geq 3$, can be found in [25].

Following Weinstein [29], we introduce a functional $\Lambda(\Theta)$ by

$$\Lambda(\Theta) = \frac{\mathsf{K}(\Theta)}{\left(\int_{-\infty}^{\infty} F(\Theta)\ dx\right)^{2/3}}, \quad \Theta = (f_1, ..., f_N, g),$$

where $f_1, ..., f_N, g \in H^1$. If $\Lambda(\Theta)$ has a critical point at $\Theta = (\phi_1, ..., \phi_N, \psi)$, then a computation of the Fréchet derivative of Λ at $(\phi_1, ..., \phi_N, \psi)$ shows that $(\phi_1, ..., \phi_N, \psi)$ is, up to a constant multiple, a solution of (1.3). Consider now the following unconstrained minimization problem

$$\min\left\{\Lambda(f_1, ..., f_N, g) : \vec{0} \neq (f_1, ..., f_N, g) \in \mathcal{Y}\right\}. \tag{P2}$$

Proposition 3.4. *The problem (P1) is equivalent to (P2). More precisely, any solution of (P1) is a minimizer of (P2), and if $\Delta = (\phi_1, ..., \phi_N, \psi)$ is a minimizer of (P2) then the rescaling*

$$(\phi_1, ..., \phi_N, \psi) \mapsto \frac{\lambda^{1/3}}{\left(\int_{-\infty}^{\infty} F(\Delta)\ dx\right)^{1/3}}\ (\phi_1, ..., \phi_N, \psi)$$

is a solution of the problem (P1). Moreover, if we define Θ by

$$\Theta = \frac{2}{3}\Lambda(\Delta)\ \frac{\Delta}{\left(\int_{-\infty}^{\infty} F(\Delta)\ dx\right)^{1/3}} = \frac{2}{3} I_1\ \frac{U}{\lambda^{1/3}},$$

then Θ does not depend on λ and solves (1.3)*, and therefore when substituted into* (1.2) *yields a travelling solitary wave for* (1.1)*.*

Proof. The proof follows from Theorem 2.1. □

Theorem 3.5. *Suppose $c_\tau > 0$ and $\sigma > 0$. There exists a non-trivial, solitary-wave solution $(\Phi_1, ..., \Phi_N, \Psi)$ of the system* (1.3) *such that*

$$\Phi_1(x) \geq 0,\ \ .\ .\ ., \Phi_N(x) \geq 0, \Psi(x) \geq 0, \tag{3.8}$$

for $x \in \mathbb{R}$. Moreover, the functions $\Phi_1(x), ..., \Phi_N(x), \Psi(x)$ can be chosen to be even, strictly positive, and non-increasing for $x \geq 0$.

Proof. Let $\Theta = (f_{01}, ..., f_{0N}, g_0)$ be a minimizer of (P2) in $\mathcal{Y}$. Define $\phi_j = |f_{0j}|, j = 1, ..., N$, and $\psi = |g_0|$. Then ϕ_j and ψ are in H^1, and a similar inequality of the form (3.5) holds for ϕ_j and ψ. Denote $\Delta = (\phi_1, ..., \phi_N, \psi)$. Then $\mathsf{K}(\Delta) \leq \mathsf{K}(\Theta)$ and

$$\int_{-\infty}^{\infty} F(\Delta)\ dx \geq \int_{-\infty}^{\infty} F(\Theta)\ dx.$$

It follows that $\Lambda(\Delta) \leq \Lambda(\Theta)$. Hence Δ is also a minimizer of (P2). Consequently, there exists a constant $a > 0$ such that such that $(\Phi_1, ..., \Phi_N, \Psi)$ defined by

$$\Phi_j = a\phi_1 = a|f_{0j}|,\ 1 \leq j \leq N,\ \text{and}\ \Psi = a\psi = a|g_0|,$$

is a solution of (1.3) and (3.8) holds. To prove $\Phi_j > 0$, observe that (2.23) implies $\alpha_j K_\sigma \star (\Psi\Phi_j) = \Phi_j$ for all $1 \leq j \leq N$, where K_σ is defined as in (3.3). It follows that $\Phi_j > 0$ on $\mathbb{R}$ for each $1 \leq j \leq N$.

Next, as above, let $\Theta = (f_{01}, ..., f_{0N}, g_0)$ be a minimizer of the problem (P2). Define $\Delta_* = (\phi_{*1}, ..., \phi_{*N}, \psi_*)$ by setting $\phi_{*j} = f_{0j}^*, 1 \leq j \leq N$, and $\psi_* = g_0^*$. Then, using the Hardy-Littlewood inequality (Theorem 3.2(ii)), we have that

$$\int_{-\infty}^{\infty} F(\Theta)\ dx \leq \int_{-\infty}^{\infty} F(\Delta_*)\ dx.$$

Using the Pólya-Szegő inequality (Theorem 3.2(iii)) and the fact that rearrangement preserves L^p norm (Theorem 3.2(i)), it follows that $\mathsf{K}(\Delta_*) \leq \mathsf{K}(\Theta)$. Hence Δ_* is also a minimizer of (P2). Then, as in the preceding paragraph, there exists $a \in \mathbb{R}$ such that $(\Phi_j, \Psi) = a\Delta_*$ is a solution of (1.3). Since $\Phi_j = a\phi_{*j}, 1 \leq j \leq N$, and $\Psi = a\psi_*$ are non-increasing functions of $|x|$, this completes the proof. $\square$

We now prove positivity of Fourier transforms of the solitary waves.

Theorem 3.6. *Suppose $c_\tau > 0$ and $\sigma > 0$. Then there exists a solution $(\Phi_1, ..., \Phi_N, \Psi)$ of (1.3) such that*

$$\widehat{\Phi}_1(\xi) \geq 0,\ .\ .\ .,\widehat{\Phi}_N(\xi) \geq 0, \widehat{\Psi}(\xi) \geq 0, \tag{3.9}$$

for $\xi \in \mathbb{R}$. Moreover, $\Phi_1, ..., \Phi_N, \Psi$ may be chosen so that $\widehat{\Phi}_1, ..., \widehat{\Phi}_N, \widehat{\Psi}$ are even, strictly positive, and non-increasing functions of $|\xi|$. It also follows that $\Phi_1, ..., \Phi_N, \Psi$ are even functions.

Proof. We follow ideas of Albert [1]. Let $\Theta = (f_{01}, ..., f_{0N}, g_0)$ be a minimizer of the problem (P2) in $\mathcal{Y}$. Choose $u_1, ..., u_N, v \in L^2$ such that

$$\widehat{u}_j = |\widehat{f}_{0j}|\ \ \text{and}\ \widehat{v} = |\widehat{g}_0|,\ j = 1, 2, ..., N.$$

The functions $u_1, ..., u_N, v$ are real-valued since $|\widehat{f}_{0j}|$ and $|\widehat{g}_0|$ are real-valued and even; and $U = (u_1, ..., u_n, v)$ belongs to $\mathcal{Y}$. Then $\mathsf{K}(U) = \mathsf{K}(\Theta)$ and using Theorem 3.3, we

have

$$\int_{-\infty}^{\infty} F(U)\ dx = \frac{1}{3}\ \beta\ \widehat{v^3}(0) + \sum_{j=1}^{N} \alpha_j\ \widehat{u_j^2 v}(0)$$

$$\geq \frac{1}{3}\ \beta\ \widehat{g_0^3}(0) + \sum_{j=1}^{N} \alpha_j\ \widehat{f_{0j}^2 g_0}(0) = \int_{-\infty}^{\infty} F(\Theta)\ dx.$$

It follows that $\Lambda(U) \leq \Lambda(\Theta)$. Hence U is also a minimizer of (P2). As noted before, there exists a constant $a > 0$ such that

$$(\Phi_1, ..., \Phi_N, \Psi) = a(u_1, ..., u_N, v)$$

is a solution of (1.3). Since $\widehat{\Phi}_j = a|\widehat{f}_{0j}| \geq 0$ and $\widehat{\Psi} = a|\widehat{g}_0| \geq 0$, the first assertion of the Theorem follows.

On the other hand, since for each $1 \leq j \leq N$, f_{0j} is real-valued,

$$\widehat{\Phi}_j(\xi) = a|\widehat{f}_{0j}(\xi)| = \frac{a}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f_{0j}(x)\cos(x\xi)\ dx - i\int_{-\infty}^{\infty} f_{0j}(x)\sin(x\xi)\ dx \right|$$

$$= \frac{a}{\sqrt{2\pi}} \sqrt{\left| \int_{-\infty}^{\infty} f_{0j}(x)\cos(x\xi)\ dx \right|^2 + \left| \int_{-\infty}^{\infty} f_{0j}(x)\sin(x\xi)\ dx \right|^2}$$

is an even function, and hence for all $1 \leq j \leq N$,

$$\Phi_j(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}_j(\xi)\cos(x\xi)\ d\xi + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}_j(\xi)\sin(x\xi)\ d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}_j(\xi)\cos(x\xi)\ d\xi$$

is an even function. Similarly, one can show that Ψ is even.

To prove positivity of the Fourier transforms of the solitary waves, let $\Theta = (f_{01}, ..., f_{0N}, g_0)$ be, as above, a minimizer of (P2). Choose $f_1, ..., f_N, g \in L^2$ such that $\widehat{f_j} = \widehat{f}_{0j}^*$, and $\widehat{g} = \widehat{g}_0^*$. Denote $\Delta = (f_1, ..., f_N, g)$. We claim that $\Lambda(\Delta) \leq \Lambda(\Theta)$. First, using statements (i) and (iii) of Theorem 3.2, it follows that $\mathsf{K}(\Delta) \leq \mathsf{K}(\Theta)$. Hence to prove the claim it suffices to show that the denominator of $\Lambda(\Delta)$ is greater than or equal to that of $\Lambda(\Theta)$. Using the inequality of F. Riesz concerning convolutions of symmetric rearrangements of functions (Theorem 3.3), we have

$$\int_{-\infty}^{\infty} F(\Delta)\ dx = \frac{\beta}{3} \left[\widehat{g} \star \widehat{g} \star \widehat{g}\right](0) + \sum_{j=1}^{N} \alpha_j \left[\widehat{f}_j \star \widehat{f}_j \star \widehat{g}\right](0)$$

$$= \frac{\beta}{3} \left[\widehat{g}_0^* \star \widehat{g}_0^* \star \widehat{g}_0^*\right](0) + \sum_{j=1}^{N} \alpha_j \left[\widehat{f}_{0j}^* \star \widehat{f}_{0j}^* \star \widehat{g}_0^*\right](0)$$

$$\geq \frac{\beta}{3} \left[\widehat{g}_0 \star \widehat{g}_0 \star \widehat{g}_0\right](0) + \sum_{j=1}^{N} \alpha_j \left[\widehat{f}_{0j} \star \widehat{f}_{0j} \star \widehat{g}_0\right](0) = \int_{-\infty}^{\infty} F(\Theta)\ dx.$$

This proves the claim. Hence Δ is also a minimizer of (P2), and it follows that there exists $a \in \mathbb{R}$ such that

$$(\Phi_1, ..., \Phi_N, \Psi) = a(f_1, ..., f_N, g)$$

is a solution of (1.3). Since $\widehat{\Phi}_j = a\widehat{f}^*_{0j}$ and $\widehat{\Psi} = a\widehat{g}^*_0$ are non-increasing functions of $|\xi|$, it remains only to show that $\widehat{\Phi}_1, ..., \widehat{\Phi}_N$, and $\widehat{\Psi}$ are everywhere positive. We only prove that $\widehat{\Phi}_1$ is everywhere positive. If this is not the case, then the support of $\widehat{\Phi}_1$ is a finite closed interval $[-a_1, a_1]$. On the other hand, the support of $\widehat{\Phi}_1 \star \widehat{\Psi}$ strictly contains $[-a_1, a_1]$, so that $(\xi^2 + \sigma)\widehat{\Phi}_1$ can not equal to $\widehat{\Psi} \star \alpha_1 \widehat{\Phi}_1$. This then contradicts the first equation in (3.1) with $j = 1$. Similarly, one can prove that $\Phi_2, ..., \Phi_N, \widehat{\Psi}$ are everywhere positive. □

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